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PROPERTIES OF INTERSECTING FAMILIES OF ORDERED SETS

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The first part of this paper deals with families of ordered k-tuples having a common element in different position. A quite general theorem is given and special cases are considered. The second part deals with t families of sets with some intersection properties, and generalizes results by Bollobás, Frankl, Lovász and Füredi to this case.

1. Introduction

Intersection theorems on families of sets have long been of interest in extremal combinatorics, ever since the celebrated theorem of Erdős, Ko and Rado [6]. In the first part of this paper we show a quite general intersection theorem, that deals with families of ordered k-tuples every two of which satisfy a certain intersection property, and show a tight bound on the size of such a family.

Another classical extremal intersection theorem, which is due to Béla Bollobás, and was given several proofs and extensions, states the following: Let $\mathcal{F}$ and $\mathcal{F}'$ be two families of sets, so that for every $A_i \in \mathcal{F}$ and $B_j \in \mathcal{F}'$, $|A_i| \le k$, $|B_j| \le l$, and $A_i \cap B_j = \emptyset$ iff $i = j$. Then $|\mathcal{F}| \le \binom{k+l}{k}$. P. Frankl [4] extended this result and showed that it remains true under considerably weaker terms. In the second part of this paper we turn to k families of sets, where we formalize and prove an analog theorem to Bollobás theorem, together with the analog to Frankl's extension.

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2. Families of ordered sets

2.1. Ordered sets with common element in a different position

In this section, we show a general theorem regarding a family of ordered sets, with the following property: Every two sets in this family share an element whose position in one set is different from that in the other. We also give some interesting examples for application of the theorem.

Let $\mathcal{F}$ be a family of ordered sets, each with k different elements. For every two sets a, b, denote by $a \prec b$ the fact that a appears before b in this family. Note that the bound we get here does not depend on n – the size of the universe.

Theorem 2.1. *Let $\mathcal{F}$ be a family of k-element ordered sets over a universe of n elements. For each index $1 \le i \le k$ (position in the set) let $L_i \subseteq [1..k] \setminus \{i\}$ be a list of indices. If for every two sets $a \prec b$ there exist i and $j \in L_i$ such that $a_i = b_j$, then $|\mathcal{F}| \le \prod_{i=1}^{k}(|L_i|+1)$.*

Proof. For each element in the universe associate a different integer from $[1..n]$. Thus, each k-tuple a is associated with a k-tuple of integers $\{r_i^a\}_{i=1}^k$. For each position $1 \le j \le k$ associate the unknown x_j. With each set $a \in \mathcal{F}$ we associate the following polynomial in k variables $x_1, x_2, \ldots, x_k$: $Q_a = \prod_{i=1}^{k} \prod_{t \in L_i} (x_t - r_i^a)$. Q_a vanishes by substituting its unknowns for the numbers associated with any k-tuple b satisfying $a \prec b$, but not by substituting it for those of a. Thus, the polynomials $\{Q_a\}$ are linearly independent over $\mathbb{R}$. Since the power of each unknown x_i varies from 0 to $|L_i|$, the polynomials $\{Q_a\}$ reside in a space of dimension $\prod_{i=1}^{k}(|L_i|+1)$. ∎

Theorem 2.1 has the following nice implications:

Corollary 2.2. *Let $\mathcal{F}$ be a family of k-elements ordered sets on a universe of n elements such that for every two sets $a \prec b$ there exists an element $a_i = b_j$ $i < j$ (i.e., these sets are intersected forward). Then $|\mathcal{F}| \le k!$.*

Proof. For every $1 \le i \le k$ let $L_i = \{i+1, \ldots, k\}$ so that $|L_i| = k - i$. By the assumptions of the corollary, for every two sets $a \prec b$, there exist i and $j \in L_i$ such that $a_i = b_j$. Thus, by Theorem 2.1, $|\mathcal{F}| \le \prod_{i=1}^{k}(|L_i|+1) = \prod_{i=1}^{k}(k-i+1) = k!$. ∎

To see that this result is tight – take all permutations of k elements.

Another corollary of Theorem 2.1 regards the case where the intersection between the sets is not restricted at all by the lists:

Corollary 2.3. *Let $\mathcal{F}$ be a family of k-elements ordered sets on a universe of n elements, such that every two sets have a common element in a different position. Then $|\mathcal{F}| \leq k^k$.*

This is the best bound known, and a probabilistic proof for it can be inferred from the work of Tuza [7]. If we take n, the size of our universe as a parameter, then we can slightly tighten this bound. Recall that $(n)_k$ is the number of possibilities to choose an ordered k-tuple out of n elements.

Proposition 2.4. *Let $\mathcal{F}$ be a family of k-elements ordered sets on a universe of n elements, such that every two sets have a common element in a different position, and let $t = n \pmod k$, $0 \leq t < k$. Then $|\mathcal{F}| \leq \frac{(n)_k}{\lceil \frac{n}{k} \rceil^t \cdot \lfloor \frac{n}{k} \rfloor^{k-t}} \approx k^k \cdot \frac{(n)_k}{n^k}$.*

Proof. Take a random permutation of $[1..n]$. Associate with each ordered set $v \in \mathcal{F}$ an event A_v being that every element in the set appears in the same place in the permutation as it does in $v \pmod k$. Let $t = n \pmod k$. Then the probability for the event A_v is $\frac{\lceil \frac{n}{k} \rceil}{n} \cdot \frac{\lceil \frac{n}{k} \rceil}{n-1} \cdots \frac{\lceil \frac{n}{k} \rceil}{n-t+1} \cdot \frac{\lfloor \frac{n}{k} \rfloor}{n-t} \cdot \frac{\lfloor \frac{n}{k} \rfloor}{n-t-1} \cdots \frac{\lfloor \frac{n}{k} \rfloor}{n-k+1}$. It is easy to see that no two such events can co-occur in the same permutation, thus there are at most $\frac{1}{p(A_v)}$ such events, that is, such k-tuples. ∎

Note that if k divides n, then $\frac{(n)_k}{\lceil \frac{n}{k} \rceil^t \cdot \lfloor \frac{n}{k} \rfloor^{k-t}} = k^k \cdot \frac{(n)_k}{n^k}$.

Proposition 2.4 is tight for $n = k+1$, giving the bound $\frac{(k+1)!}{2}$ (see [1]), but we do not know the tight bound for every n.

Our third corollary illustrates yet another way to apply Theorem 2.1:

Corollary 2.5. *Let $\mathcal{F}$ be a family of k-elements ordered sets on a universe of n elements, such that for every two sets satisfying $a \prec b$, there exists a common element $a_i = b_j$ so that $0 < |i-j| \leq l$ (i.e, a "local" intersection). Then $|\mathcal{F}| \leq (2l+1)^k$.*

Remark. Corollary 2.5 remains true if the distance $|i-j|$ is taken cyclically, i.e., the distance between two elements is the minimum between the two possible ones.

2.2. Families of ordered sets every two of which disagree

We say that two ordered sets a, b *disagree* with each other, if there exist $i_1 < i_2$ and $j_1 < j_2$, such that $a_{i_1} = b_{j_2}$ and $a_{i_2} = b_{j_1}$. That is to say, there are two elements that appear both in a and in b, but in a different order.

Theorem 2.6. *Let $\mathcal{F}$ be a family of k-elements ordered sets for which every two sets disagree. Then $|\mathcal{F}| \leq k!$.*

Proof. Assume all in all there are n elements. Pick a permutation of them uniformly at random. Clearly the probability of every ordered set a to come up in the "right" order is $(k!)^{-1}$. But if a comes out in the right order, no other set does, due to the disagreement between them. Thus, all these events are mutually disjoint, and therefore there can be at most $k!$ of them. ∎

2.3. Ordered set of sets and a generalization of the skew/non-skew versions of Bollobás' theorem

In this section, we generalize Theorem 2.6 and Corollary 2.2 in a way that also generalizes Bollobás' theorem [3] and its skew version.

Theorem 2.6a (Bollobás' theorem for k families of sets). *Let $l_1, l_2, \ldots, l_k$ be positive integers. Consider the following matrix of sets:*

$$\begin{matrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{matrix}$$

If these sets satisfy:

1. *for every $1 \leq j \leq k$, $1 \leq i \leq m$, $|A_{ij}| \leq l_j$;*
2. *for every fixed i, all sets A_{ij} are pairwise disjoint;*
3. *for each $i < i'$ there exist some $j_1 < j_2$ and $j'_1 < j'_2$ so that $A_{i,j_1} \cap A_{i',j'_2} \neq \emptyset$ and $A_{i,j_2} \cap A_{i',j'_1} \neq \emptyset$;*

then $m \leq \frac{(\sum_{r=1}^{k} l_r)!}{\prod_{r=1}^{k} l_r!}$.

The proof of this theorem is by a straightforward extension of the argument in the proof of Theorem 2.6, by considering the probability that all k sets in a row of the matrix appear in the right order.

We now generalize a theorem by Lovász, in a way that will enable us to derive a generalization of Corollary 2.2 to k-tuples of sets. The proof here is a little bit more involved.

Theorem 2.7 (skew version of Bollobás' theorem for k families of subspaces). *Let $l_1, l_2, \ldots, l_k$ be positive integers. Let U be a linear space* *over a field $\mathbb{F}$. Consider the following matrix of subspaces:*

$$\begin{matrix} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{matrix}$$

If these subspaces satisfy:

1. *for every $1 \le j \le k$, $1 \le i \le m$, $\dim U_{ij} \le l_j$;*
2. *for every fixed i, all subspaces U_{ij} are pairwise disjoint;*
3. *for each $i < i'$ there exists some $j < j'$ such that $U_{ij} \cap U_{i'j'} \neq \emptyset$;*

then $m \le \frac{(\sum_{r=1}^{k} l_r)!}{\prod_{r=1}^{k} l_r!}$.

Proof. Assume w.l.o.g. that $\dim U_{i,j}$ is exactly l_j for every i, j, and that the field $\mathbb{F}$ is infinite. Let $t_j = \sum_{j'=j}^{k} l_{j'}$, and let $n = \dim U$. Obviously, $n \ge \sum_{j=1}^{k} l_j$.

Construction. Let $V = V_1 \bigoplus V_2 \bigoplus V_3 \bigoplus \cdots \bigoplus V_{k-1}$ be a $\sum_{j=1}^{k-1} t_j$ dimensional linear space over $\mathbb{F}$, decomposed into the direct sum of subspaces V_j, each of dimension t_j. In addition, for every $1 \le i \le k-1$, let $T_i : U \to V_i$ be a linear transformation with the following property: For every $1 \le o, q \le k$ and $i \le p, r \le k$, $\dim(T_i(U_{op})) = \dim(U_{op})$ and $\dim(U_{op} \cap U_{qr}) = \dim(T_i(U_{op}) \cap T_i(U_{qr}))$. Such a transformation exists by corollary 3.14 in [2]. Finally, for every row i in the matrix, let $v_i = \bigwedge_{j=1}^{k-1} \wedge T_j(U_{ij})$, and let $w_i = \bigwedge_{j=1}^{k-1} \bigwedge_{r=j+1}^{k} \wedge T_j(U_{ir})$.

Claim. For every $i \le i'$, $v_i \wedge w_{i'} \neq 0$ iff $i = i'$. Proof of claim: Assume $i \neq i'$, that is, $i < i'$. Then, by the theorem conditions, there exists $j < j'$ for which $U_{ij} \cap U_{i'j'} \neq 0$. According to the way we chose the transformations, this is true iff $T_q(U_{ij}) \cap T_q(U_{i'j'}) \neq 0$ for every $1 \le q \le j$. Since $v_i \wedge w_{i'}$ contains both $T_j(U_{ij})$ and $T_j(U_{i'j'})$ as factors, we get that $v_i \wedge w_{i'} = 0$. If however $i = i'$, then $v_i \wedge w_{i'}$ is the wedge product of only disjoint subspaces, thus $v_i \wedge w_{i'} \neq 0$.

Using the claim above and the triangular criterion, we get that the vectors v_i are linearly independent in the space $\wedge_{i=1}^{k-1} V_i$, whose dimension is $\prod_{i=1}^{k-1} \binom{t_i}{l_i} = \frac{(\sum_{r=1}^{k} l_r)!}{\prod_{r=1}^{k} l_r!}$. Thus, there are at most this number of vectors v_i. Thus, $m \le \frac{(\sum_{r=1}^{k} l_r)!}{\prod_{r=1}^{k} l_r!}$. ∎

It is now straightforward to obtain our second main result, and generalize Corollary 2.2 to k-tuples of sets. This also generalizes a famous result by P. Frankl [4].

Theorem 2.8 (skew version of Bollobás' theorem for k families of sets). *Let $l_1, l_2, \dots, l_k$ be positive integers. Consider the following matrix of sets:*

$$\begin{matrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \dots & \dots & \vdots & \dots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{matrix}$$

If these sets satisfy:

1. *for every $1 \le j \le k$, $1 \le i \le m$, $|A_{ij}| \le l_j$;*
2. *for every fixed i, all sets A_{ij} are pairwise disjoint;*
3. *for each $i < i'$ there exists some $j < j'$ such that $A_{ij} \cap A_{i'j'} \ne \emptyset$;*

then $m \le \frac{(\sum_{r=1}^{k} l_r)!}{\prod_{r=1}^{k} l_r!}$.

Proof. Let $X = \bigcup_{i,j} A_{ij}$ be our universe of elements, and let $|X| = n$. With each element x, we associate one of the standard basis vectors e_x of $\mathbb{R}^n$, and with each set A_{ij} we associate the subspace $sp\{e_x |\ x \in A_{ij}\}$. These subspaces meet the criteria of Theorem 2.7, and the bound follows. ∎

The bounds of Theorems 2.6a and 2.8 are tight, as can be seen from partitioning a ground set of size $\sum_{j=1}^{k} l_j$ into sets of size l_j in all possible ways.

Note that these two bounds are the same, as is the case for two families of sets. Although for two families of sets Frankl's theorem implies Bollobás' theorem, this is not the case here, so we formalized Bollobás' theorem for k sets independently.

To complete the picture, we will end by proving the threshold version of Theorem 2.8. This generalizes a result by Füredi [5].

Lemma 2.9. *Let $l_1, l_2, \dots, l_k$ be positive integers. Let U be a linear space over a field $\mathbb{F}$. Consider the following matrix of subspaces:*

$$\begin{matrix} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \dots & \dots & \vdots & \dots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{matrix}$$

If these subspaces satisfy:

1. *for every $1 \le j \le k$, $1 \le i \le m$, $\dim U_{ij} \le l_j$;*
2. *for every i, j and j', $\dim(U_{ij} \cap U_{ij'}) \le t$;*

3. *for each* $i<i'$ *there exists some* $j<j'$ *such that* $\dim(U_{ij}\cap U_{i'j'})>t$;

then $m\leq\frac{[(\sum_{r=1}^{k}l_r)-kt]!}{\prod_{r=1}^{k}(l_r-t)!}$.

Proof. By reduction to Theorem 2.8. Let U_0 be a subspace of U, such that $\dim(U_0)=\dim(U)-t$ and in addition, for every i,j,p,q, U_0 is in general position with regard to U_{ij} and to $U_{ij}\cap U_{pq}$. Under these conditions, it can be easily seen that:

1. For every $i<i'$, there exist some $1\leq j<j'\leq k$ for which $\dim(U_{ij}\cap U_{i'j'}\cap U_0)\geq 1$.
2. For every i,j,j', $j\neq j'$, $U_{ij}\cap U_{ij'}\cap U_0=0$.
3. For every i,j, $\dim(U_{ij}\cap U_0)=l_j-t$.

Thus, the conditions of Theorem 2.8 are met for the matrix $U_{ij}\cap U_0$, and accordingly, $m\leq\frac{[(\sum_{r=1}^{k}l_r)-kt]!}{\prod_{r=1}^{k}(l_r-t)!}$. ∎

The appropriate sets theorem would be this:

Theorem 2.10. *Let* $l_1,l_2,\dots,l_k$ *be positive integers. Consider the following matrix of sets:*

$$\begin{matrix} A_{11} & A_{12} & \cdots & A_{1k}\\ A_{21} & A_{22} & \cdots & A_{2k}\\ \dots & \dots & \vdots & \dots\\ A_{m1} & A_{m2} & \cdots & A_{mk}\end{matrix}$$

If these sets satisfy:

1. *for every* $1\leq j\leq k$, $1\leq i\leq m$, $\dim U_{ij}\leq l_j$;
2. *for every* i, j *and* j', $|A_{ij}\cap A_{ij'}|\leq t$;
3. *for each* $i<i'$ *there exists some* $j<j'$ *such that* $|A_{ij}\cap A_{i'j'}|>t$;

then $m\leq\frac{[(\sum_{r=1}^{k}l_r)-kt]!}{\prod_{r=1}^{k}(l_r-t)!}$.

Proof. We derive the theorem from Lemma 2.9 in the very same way Theorem 2.8 was derived from Theorem 2.7. Namely, if $X=\bigcup_{i,j}A_{ij}$ is our universe of elements, and $|X|=n$, associate with each element x one of the standard basis vectors e_x of $\mathbb{R}^n$, and with each set A_{ij} the subspace $\mathrm{span}\{e_x|x\in A_{ij}\}$. These subspaces meet the criteria of Theorem 2.9, and the bound follows. ∎

Proposition 2.10 is tight as well, as can be seen by taking the ground set to be of size $t+\sum_{j=1}^{k}(l_j-t)$, setting a constant set B of size t to be a subset of every A_{ij}, and then partition the remaining $\sum_{j=1}^{k}(l_j-t)$ elements in all possible ways to complete A_{ij} to size l_j for every $1\leq j\leq k$.

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